

ON THE CONSTANCY REGIONS FOR MIXED TEST IDEALS

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ABSTRACT. In this note we study the partition of $\mathbb{R}_{\geq 0}^n$ given by the regions where the mixed test ideals $\tau(\mathfrak{a}_1^{t_1} \dots \mathfrak{a}_n^{t_n})$ are constant. We show that each region can be described as the preimage of a natural number under a p -fractal function $\varphi : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{N}$. In addition, we give some examples illustrating that these regions do not need to be composed of finitely many rational polytopes.

1. INTRODUCTION

In this note, we study the dependence of mixed test ideals on parameters, and show that the emerging picture is quite different from that in the case of mixed multiplier ideals in characteristic zero.

Multiplier ideals have been intensively studied over the last two decades, as they play an important role in birational geometry, see for example [Laz]. Given a smooth complex variety X and a nonzero ideal sheaf \mathfrak{a} , one can define for any parameter $c > 0$ an ideal $\mathcal{J}(\mathfrak{a}^c)$, called multiplier ideal. This ideal is described via a log resolution $\pi : X' \rightarrow X$ of the pair (X, \mathfrak{a}) , i.e. a proper birational map, with X' smooth, and such that $\mathfrak{a}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$, where E is a simple normal crossing divisor. Then,

$$(1.1) \quad \mathcal{J}(\mathfrak{a}^c) := \pi_* \mathcal{O}(K_{X'/X} - \lfloor cE \rfloor),$$

where $K_{X'/X}$ is the relative canonical divisor.

Mixed multiplier ideals extend the previous definition to the case of several ideals: for nonzero ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ and positive numbers c_1, \dots, c_n we take a log resolution for the pair $(X, \mathfrak{a}_1 \cdots \mathfrak{a}_n)$ and set the mixed multiplier ideal to be

$$\mathcal{J}(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}) := \pi_* \mathcal{O}(K_{X'/X} - \lfloor c_1 E_1 + \dots + c_n E_n \rfloor),$$

where $\mathcal{O}_{X'}(-E_i) = \mathfrak{a}_i \mathcal{O}_{X'}$.

Test ideals were introduced by Hara and Yoshida in [HY] as an analogue of multiplier ideals in positive characteristic. One question that was studied since [HY] is which properties of multiplier ideals have analogues for test ideals. For example, for multiplier ideals the *jumping numbers* of \mathfrak{a} are defined as the positive real numbers c such that $\mathcal{J}(\mathfrak{a}^c) \neq \mathcal{J}(\mathfrak{a}^{c-\epsilon})$ for every $\epsilon > 0$ (cf. [ELSV]). It is easy to see from the definition (1.1) that for each \mathfrak{a} these numbers are discrete and rational. Thus it was expected that this was the case also in positive characteristic. Blickle, Mustařă and Smith proved discreteness and rationality of the analogous positive characteristic invariants in [BMS], but the proof was more involved.

In the mixed multiplier ideal setting, it follows from the above description in terms of a log resolution that for every b_1, \dots, b_n the region

$$\{(c_1, \dots, c_n) \in \mathbb{R}_{\geq 0}^n \mid c_i \leq b_i \text{ for all } i\}$$

can be decomposed in a finite set of rational polytopes with nonoverlapping interiors, such that on the interior of each face of each polytope the mixed multiplier ideal $\mathcal{J}(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n})$ is constant. It was expected that in the positive characteristic setting we would have a similar picture.

In the present note we prove that this is not the case, but we can still get a nice decomposition. This decomposition depends on a p -fractal function, that is, a function $\varphi : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{N}$ satisfying the following property. If we restrict φ to a bounded domain D , then the vector space generated by the functions $\phi(t_1, \dots, t_n) = \varphi((t_1+b_1)/p^e, \dots, (t_n+b_n)/p^e)$ with b_i integers and $((t_1+b_1)/p^e, \dots, (t_n+b_n)/p^e) \in D$, is finite dimensional (Definition 4.1). Explicitly, we show:

Theorem. [Theorem 4.6] *For an F -finite, regular ring R essentially of finite type over a field of positive characteristic and non zero ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ of R , there is a p -fractal function $\varphi : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{N}$ such that*

$$\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}) = \tau(\mathfrak{a}_1^{d_1} \cdots \mathfrak{a}_n^{d_n}) \iff \varphi(c_1, \dots, c_n) = \varphi(d_1, \dots, d_n),$$

and therefore the constancy regions are of the form $\varphi^{-1}(i)$ for $i \in \mathbb{N}$.

Roughly speaking, this shows that each constancy region has a p -fractal structure that, as we see in the examples in Section 5, can be intricate.

This note is structured as follows. In section 2 we recall the definition of test ideals and mixed test ideals following [BMS] and state some of the theorems that were proved there. In section 3 we give our main definitions and deduce some basic consequences of these definitions. We prove our main theorem in section 4. In the last section, we give an example of a constancy region that is not a finite union of polyhedral regions.

ACKNOWLEDGMENTS

I would like to thank Mircea Mustaa for introducing me to this problem and for guiding my work. I would also like to thank Anglica Benito and Luis Nuez for comments and suggestions on earlier drafts. Finally, I would like to thank Rui Huang for doing the graphs in section 5.

2. PRELIMINARIES

Recall that a ring R of positive characteristic is F -finite if the Frobenius morphism $F : R \rightarrow R$ is finite. Throughout this note we let R be a regular ring essentially of finite type over an F -finite field k of positive characteristic p . In particular, R is F -finite as well. We now recall the basic definitions and properties related to test ideals and refer to [BMS] for proofs and details.

Given an ideal \mathfrak{b} in R , we denote by $\mathfrak{b}^{[1/p^e]}$ the smallest ideal \mathfrak{J} such that $\mathfrak{b} \subseteq \mathfrak{J}^{[p^e]} := (f^{p^e} | f \in \mathfrak{J})$. The existence of a smallest such ideal is a consequence of the flatness of the Frobenius map in the regular case. The following proposition gives an explicit description of $\mathfrak{b}^{[1/p^e]}$ when R is free over R^{p^e} .

Proposition 2.1. [BMS, Proposition 2.5] *Suppose that R is free over R^q , for $q = p^e$, and let e_1, \dots, e_N be a basis of R over R^q . If h_1, \dots, h_n are generators of an ideal \mathfrak{b} of R , and if for every $i = 1, \dots, n$ we write*

$$h_i = \sum_{j=1}^N a_{i,j}^q e_j$$

with $a_{i,j} \in R$, then

$$\mathfrak{b}^{[1/p^e]} = (a_{i,j} | i \leq n \text{ and } j \leq N).$$

Test ideals were introduced by Hochster and Huneke [HH] as a tool in their tight closure theory, and were later generalized by Hara and Yoshida [HY] in the context of pairs (R, \mathfrak{a}^c) , where \mathfrak{a} is an ideal in R and c is a real parameter. Blickle, Mustařă, and Smith [BMS] gave an elementary description of these ideals in the case of a regular F -finite ring R . It is this description which we take as our definition.

Definition 2.2. Given a non-negative number c and a nonzero ideal \mathfrak{a} , we define the *generalized test ideal of \mathfrak{a} with exponent c* to be

$$\tau(\mathfrak{a}^c) = \bigcup_{e>0} (\mathfrak{a}^{\lceil cp^e \rceil})^{[1/p^e]},$$

where $\lceil c \rceil$ stands for the smallest integer $\geq c$.

The ideals in the above union form an increasing chain of ideals; therefore as R is Noetherian, they stabilize. Hence for e large enough $\tau(\mathfrak{a}^c) = (\mathfrak{a}^{\lceil cp^e \rceil})^{[1/p^e]}$. In the principal ideal case we can say more.

Proposition 2.3. [BMS2, Lemma 2.1] *If $\lambda = \frac{m}{p^e}$ for some positive integer m , then $\tau(f^\lambda) = (f^m)^{[1/p^e]}$.*

It can be shown that as the parameter c varies over the reals, only countably many different test ideals appear; moreover, we have:

Theorem 2.4. [BMS, Proposition 2.14] *For every nonzero ideal \mathfrak{a} and every non-negative number c , there exists $\epsilon > 0$ such that $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^{c'})$ for $c < c' < c + \epsilon$.*

Definition 2.5. A positive real number c is an *F -jumping exponent* of \mathfrak{a} if $\tau(\mathfrak{a}^c) \neq \tau(\mathfrak{a}^{c-\epsilon})$ for all $\epsilon > 0$

The F -jumping exponents of an ideal \mathfrak{a} form a discrete set of rational numbers, that is, there are no accumulation points of this set. In fact, they form a sequence with limit infinity (see [BMS, Theorem 3.1]).

As in the case of one ideal, one can define the mixed test ideal of several ideals as follows.

Definition 2.6. Given nonzero ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ of R and non-negative real numbers c_1, \dots, c_n , we define the *mixed generalized test ideal with exponents c_1, \dots, c_n* as:

$$\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}) = \bigcup_{e>0} (\mathfrak{a}_1^{\lceil c_1 p^e \rceil} \cdots \mathfrak{a}_n^{\lceil c_n p^e \rceil})^{[1/p^e]}.$$

As in the case of $\tau(\mathfrak{a}^c)$, we have $\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}) = (\mathfrak{a}_1^{\lceil c_1 p^e \rceil} \cdots \mathfrak{a}_n^{\lceil c_n p^e \rceil})^{[1/p^e]}$ for all e large enough.

Theorem 2.7. *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be nonzero ideals in the polynomial ring $R = k[x_1, \dots, x_r]$, and let $c_1 = r_1/p^s, \dots, c_n = r_n/p^s$ be such that r_1, \dots, r_n are natural numbers. If each \mathfrak{a}_i can be generated by polynomials of degree at most d , then the ideal $\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n})$ can be generated by polynomials of degree at most $\lfloor d(c_1 + \dots + c_n) \rfloor$. Here $\lfloor r \rfloor$ stands for the biggest integer $\leq r$.*

Proof. We argue as in [BMS, Proposition 3.2], where the result was proven for the case of one ideal. We know that R is free over R^{p^e} with basis

$$\{\beta_j x_1^{\alpha_1} \cdots x_r^{\alpha_r} | 0 \leq \alpha_i < p^e \text{ and } \beta_j \text{ part of a basis for } k \text{ over } k^{p^e}\}.$$

The ideal $\mathfrak{a}_1^{[p^e c_1]} \cdots \mathfrak{a}_n^{[p^e c_n]}$ can be generated by polynomials of degree at most $d[p^e c_1] + \dots + d[p^e c_n]$. Hence taking $e > s$ large enough by Proposition 2.1 the ideal

$$\tau(\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}) = (\mathfrak{a}_1^{[p^e c_1]} \cdots \mathfrak{a}_n^{[p^e c_n]})^{[1/p^e]}$$

is generated by polynomials of degree at most $(d[p^e c_1] + \dots + d[p^e c_n])/p^e = (dp^{e-s}r_1 + \dots + dp^{e-s}r_n)/p^e = d(r_1 + \dots + r_n)$. \square

3. SOME SETS ASSOCIATED TO MIXED TEST IDEALS

In this section we introduce the definitions needed for our study of mixed test ideals, and derive some basic properties. Recall that R denotes a regular ring essentially of finite type over an F -finite field k of positive characteristic.

Remark 3.1. In order to simplify notation we denote $\mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_n^{c_n}$ by $\mathfrak{a}^{\mathbf{c}}$, where $\mathbf{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_n)$, $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}_{\geq 0}^n$. We similarly denote the vector $([r_1], \dots, [r_n])$ by $[\mathbf{r}]$, where $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}_{\geq 0}^n$.

Definition 3.2. Given nonzero ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$, and I in R , we define

$$V^I(\mathbf{a}, p^e) = \left\{ \frac{1}{p^e} \mathbf{c} = \left(\frac{c_1}{p^e}, \dots, \frac{c_n}{p^e} \right) \in \frac{1}{p^e} \mathbb{Z}_{\geq 0}^n \mid \mathfrak{a}^{\mathbf{c}} \not\subseteq I^{[p^e]} \right\}$$

and

$$B^I(\mathbf{a}, p^e) = \bigcup [0, l_1] \times \dots \times [0, l_n] \subset \mathbb{R}^n,$$

where the union runs over all $(l_1, \dots, l_n) \in V^I(\mathbf{a}, p^e)$.

From this definition it follows that if $e' \geq e$ then $V^I(\mathbf{a}, p^e) \subseteq V^I(\mathbf{a}, p^{e'})$ and $B^I(\mathbf{a}, p^e) \subseteq B^I(\mathbf{a}, p^{e'})$. Indeed, if $\mathfrak{a}^{\mathbf{c}} \not\subseteq I^{[p^e]}$, then there is an element $f \in \mathfrak{a}^{\mathbf{c}}$ with $f \notin I^{[p^e]}$, and by the flatness of the Frobenius morphism we get $f^{p^{e'-e}} \in \mathfrak{a}^{p^{e'-e}\mathbf{c}}$ but $f^{p^{e'-e}} \notin I^{[p^{e'}]}$. Therefore $\mathfrak{a}^{p^{e'-e}\mathbf{c}} \not\subseteq I^{[p^{e'}]}$, hence we get the first inclusion. The second one is then straightforward.

Definition 3.3. Let $B^I(\mathbf{a}) = \bigcup_{e>0} B^I(\mathbf{a}, p^e)$ and define $\chi_{\mathbf{a}}^I : \mathbb{R}^n \rightarrow \mathbb{N}$ to be the characteristic function of the set $B^I(\mathbf{a})$. That is, $\chi_{\mathbf{a}}^I(\mathbf{c})$ is 1 if \mathbf{c} is in $B^I(\mathbf{a})$ and it is 0 otherwise.

In order to study the sets $B^I(\mathbf{a})$ it is crucial to understand how they intersect any increasing path. This motivates the following definition.

Definition 3.4. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$, and $I \neq R$ be nonzero ideals as before and let $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{Z}_{\geq 0}^n$ be such that $\mathfrak{a}^{\mathbf{r}} \subseteq \text{rad}(I)$. We denote

$$V_{\mathbf{r}}^I(\mathbf{a}, p^e) = \max\{m \in \mathbb{Z}_{\geq 0} \mid \mathfrak{a}^{m\mathbf{r}} \not\subseteq I^{[p^e]}\}.$$

Remark 3.5. While in the definition of $V^I(\mathbf{a}, p^e)$ one does not require any relation between \mathbf{a} and I , observe that we require that $\mathfrak{a}^{\mathbf{r}} \subseteq \text{rad}(I)$ when we consider $V_{\mathbf{r}}^I(\mathbf{a}, p^e)$.

Note that if $\mathfrak{a}^{mr} \not\subseteq I^{[p^e]}$ then $a^{pmr} \not\subseteq I^{[p^{e+1}]}$. Therefore $pV_{\mathbf{r}}^I(\mathfrak{a}, p^e) \leq V_{\mathbf{r}}^I(\mathfrak{a}, p^{e+1})$, hence

$$(3.1) \quad \left(\frac{V_{\mathbf{r}}^I(\mathfrak{a}, p^e)}{p^e} \right)_{e \geq 1}$$

is a non-decreasing sequence.

Proposition 3.6. *The sequence 3.1 is bounded, hence it has a limit.*

Proof. If $\mathfrak{a}^{\mathbf{r}}$ is generated by s elements, then $\mathfrak{a}^{(s(p^e-1)+1)\mathbf{r}} \subseteq (\mathfrak{a}^{\mathbf{r}})^{[p^e]}$. For l large enough such that $\mathfrak{a}^{l\mathbf{r}} \subseteq I$, we have $V_{\mathbf{r}}^I(\mathfrak{a}, p^e) \leq l(s(p^e-1)+1)-1$ for all e . Therefore $V_{\mathbf{r}}^I(\mathfrak{a}, p^e)/p^e \leq ls$, thus the sequence is bounded. \square

Definition 3.7. We call this limit the *F-threshold of \mathfrak{a} associated to I in direction $\mathbf{r} = (r_1, \dots, r_n)$* , and we denote it by $C_{\mathbf{r}}^I(\mathfrak{a})$.

Remark 3.8. In the case $n = 1$ we recover the usual definition of *F-threshold* [MTW], [BMS, Section 2.5].

Lemma 3.9. *Let $\frac{1}{p^e}\mathbf{b} = (\frac{b_1}{p^e}, \dots, \frac{b_n}{p^e})$ and $\frac{1}{p^{e'}}\mathbf{c} = (\frac{c_1}{p^{e'}}, \dots, \frac{c_n}{p^{e'}})$ be two elements in $\mathbb{R}_{\geq 0}^n$. If $\frac{b_i}{p^e} \leq \frac{c_i}{p^{e'}}$, for every i , and $e' \leq e$ then $(\mathfrak{a}^{\mathbf{c}})^{[1/p^{e'}]} \subseteq (\mathfrak{a}^{\mathbf{b}})^{[1/p^e]}$.*

Proof. It follows as in [BMS, Lemma 2.8]. The condition $b_i \leq c_i p^{e-e'}$ implies that $\mathfrak{a}_i^{b_i} \supseteq \mathfrak{a}_i^{c_i p^{e-e'}}$ for every i . Therefore

$$(\mathfrak{a}^{\mathbf{b}})^{[1/p^e]} \supseteq (\mathfrak{a}^{p^{e-e'}\mathbf{c}})^{[1/p^e]} \supseteq (\mathfrak{a}^{\mathbf{c}})^{[1/p^{e'}]}.$$

\square

Proposition 3.10. *Given any $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}_{\geq 0}^n$, there is $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}_{> 0}^n$ such that for every $\mathbf{r} = (r_1, \dots, r_n)$ with $0 < r_i < \epsilon_i$, we have $\tau(\mathfrak{a}^{\mathbf{c}}) = \tau(\mathfrak{a}^{\mathbf{c}+\mathbf{r}})$.*

Proof. We argue as in the proof of [BMS, Proposition 2.14]. We first show that there is a vector $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$, with $\epsilon_i > 0$ for all i , such that for all vectors $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{Z}^n$ with $c_i < \frac{1}{p^e}r_i < c_i + \epsilon_i$ we have that $(\mathfrak{a}^{\mathbf{r}})^{[1/p^e]}$ is constant. Indeed, otherwise there are sequences $\mathbf{r}_m = (r_{m,1}, \dots, r_{m,n}) \in \mathbb{Z}_{\geq 0}^n$ and $e_m \in \mathbb{Z}_{\geq 0}$ such that $\frac{1}{p^{e_m}}\mathbf{r}_m$ converges to \mathbf{c} , $\left(\frac{1}{p^{e_m}}r_{m,i}\right)_m$ is a decreasing sequence for every i , $e_m \leq e_{m+1}$, and $(\mathfrak{a}^{\mathbf{r}_m})^{[1/p^{e_m}]} \neq (\mathfrak{a}^{\mathbf{r}_{m+1}})^{[1/p^{e_{m+1}}]}$. It follows from Lemma 3.9 that $(\mathfrak{a}^{\mathbf{r}_m})^{[1/p^{e_m}]} \subsetneq (\mathfrak{a}^{\mathbf{r}_{m+1}})^{[1/p^{e_{m+1}}]}$ for all m , but this contradicts the fact that R is Noetherian.

Assume now that $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$ is as above and let $I = (\mathfrak{a}^{\mathbf{r}})^{[1/p^e]}$ for all $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{Z}^n$ with $c_i < \frac{1}{p^e}r_i < c_i + \epsilon_i$. We show that $I = \tau(\mathfrak{a}^{\mathbf{c}})$. Take e large enough such that $\tau(\mathfrak{a}^{\mathbf{c}}) = (\mathfrak{a}^{[p^e\mathbf{c}]})^{[1/p^e]}$ and $\frac{[p^e c_i]}{p^e} < c_i + \epsilon_i$ for every i . If all $p^e c_i$ are non-integers then $\frac{[p^e c_i]}{p^e} > c_i$ and $\tau(\mathfrak{a}^{\mathbf{c}}) = I$. Let us suppose that $p^e c_i$ is an integer precisely when $i = i_1, \dots, i_l$. Let $\mathbf{d} = (d_1, \dots, d_n)$ be the vector whose i_j coordinates are 1 and all the other are 0. As e is arbitrarily large we may also assume that $c_i < c_i + \frac{1}{p^e}d_i < c_i + \epsilon_i$ for all $i \in \{i_1, \dots, i_l\}$, hence $I = (\mathfrak{a}^{[p^e\mathbf{c}]+\mathbf{d}})^{[1/p^e]} \subseteq (\mathfrak{a}^{[p^e\mathbf{c}]})^{[1/p^e]} = \tau(\mathfrak{a}^{\mathbf{c}})$.

The reverse inclusion follows by showing $\mathfrak{a}^{[p^e\mathbf{c}]} \subseteq I^{[p^e]}$. Let $u \in \mathfrak{a}^{[p^e\mathbf{c}]}$. If $e' > e$ and e' is large enough, then $c_i < c_i + \frac{1}{p^{e'}} < c_i + \epsilon_i$, hence $\mathfrak{a}^{[p^{e'}\mathbf{c}]+1} \subseteq I^{[p^{e'}]}$. Here

$\mathbf{1}$ denotes the vector whose coordinates are all 1. Thus, for v a nonzero element in $\mathfrak{a}_1 \cdots \mathfrak{a}_n$ we have

$$vu^{p^{e'-e}} \in \mathfrak{a}^{p^{e'-e} \lceil p^e c \rceil + 1} \subseteq \mathfrak{a}^{\lceil p^{e'} c \rceil + 1} \subseteq (I^{[p^e]})^{[p^{e'-e}]}$$

This implies that u is in the tight closure of $I^{[p^e]}$, but as R is a regular ring, the tight closure of $I^{[p^e]}$ is equal to $I^{[p^e]}$ (see [HH]). This gives $\mathfrak{a}^{\lceil p^e c \rceil} \subseteq I^{[p^e]}$ hence, by definition, $\tau(\mathfrak{a}^c) = (\mathfrak{a}^{\lceil p^e c \rceil})^{[1/p^e]} \subseteq I$. \square

Definition 3.11. A positive real number c is called an *F-jumping number* of \mathfrak{a} in the direction $\mathbf{r} \neq \mathbf{0} \in \mathbb{Z}_{\geq 0}^n$, if c is such that $\tau(\mathfrak{a}^{c\mathbf{r}}) \neq \tau(\mathfrak{a}^{(c-\epsilon)\mathbf{r}})$ for every real number $\epsilon > 0$.

Proposition 3.12. If $r \in \mathbb{Z}_{\geq 0}^n$ and $\lambda \in \mathbb{R}_{\geq 0}$, then

$$\tau(\mathfrak{a}^{\lambda r_1} \cdots \mathfrak{a}^{\lambda r_n}) = \tau(\mathfrak{J}^\lambda),$$

where $\mathfrak{J} = \mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_n^{r_n}$.

Proof. By Proposition 3.10, we may assume $\lambda = \frac{s}{p^{e'}}$ with $s \in \mathbb{Z}_{\geq 0}$. For e sufficiently large, we have

$$\begin{aligned} \tau(\mathfrak{a}^{\lambda r_1} \cdots \mathfrak{a}^{\lambda r_n}) &= (\mathfrak{a}^{\lceil \lambda r_1 p^e \rceil} \cdots \mathfrak{a}^{\lceil \lambda r_n p^e \rceil})^{[1/p^e]} = (\mathfrak{a}^{sr_1 p^{e-e'}} \cdots \mathfrak{a}^{sr_n p^{e-e'}})^{[1/p^e]} \\ &= ((\mathfrak{a}^{r_1} \cdots \mathfrak{a}^{r_n})^{sp^{e-e'}})^{[1/p^e]} = ((\mathfrak{a}^{r_1} \cdots \mathfrak{a}^{r_n})^{\lambda p^e})^{[1/p^e]} = \tau(\mathfrak{J}^\lambda). \end{aligned}$$

\square

Corollary 3.13. The *F-threshold* of \mathfrak{a} associated to I in the direction $\mathbf{r} = (r_1, \dots, r_n)$ is equal to the *F-threshold* of $\mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_n^{r_n}$ associated to I .

Corollary 3.14. The set of *F-jumping numbers* of \mathfrak{a} in direction \mathbf{r} is equal to the set of *F-jumping numbers* of $\mathfrak{a}^{\mathbf{r}}$.

Therefore [BMS, Corollary 2.30] implies the following.

Corollary 3.15. The set of *F-jumping numbers* of \mathfrak{a} in the direction \mathbf{r} is equal to the set of *F-thresholds* of \mathfrak{a} , associated to various ideals I , in the direction \mathbf{r} .

Given l_1, \dots, l_n positive real numbers we denote by $[\mathbf{0}, \mathbf{l}]$ the set $[0, l_1] \times \cdots \times [0, l_n]$.

Proposition 3.16. Given nonzero ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ of R , the set $\{\tau(\mathfrak{a}^c) \mid c \in [\mathbf{0}, \mathbf{l}]\}$ is finite.

Proof. Since R is assumed to be essentially of finite type over k , arguing as in the proof of [BMS, Theorem 3.1], one can see that the assertion for all such R follows if we know it for $R = k[x_1, \dots, x_r]$, with $r \geq 1$. We will therefore assume that we are in this case.

We first show this when $k = \mathbb{F}_p$.

By Lemma 3.10, we may assume that $\mathbf{c} = (\frac{\alpha_1}{p^e}, \dots, \frac{\alpha_n}{p^e})$ with $\alpha_i \in \mathbb{N}$ and $e \geq 1$. Let d be an upper bound for the degrees of the generators of \mathfrak{a}_i , for all i . By Theorem 2.7 we have that $\tau(\mathfrak{a}^c)$ is generated by polynomials of degree $\leq ndL$, where $L = \max\{l_i\}$. Since \mathbb{F}_p is finite, there are only finitely many sets consisting of polynomials of bounded degree and therefore only finitely many ideals $\tau(\mathfrak{a}^c)$ where $c \in [\mathbf{0}, \mathbf{l}]$. Thus we have the result when R is essentially of finite type over \mathbb{F}_p .

Now if $R = k[x_1, \dots, x_r]$, let $\{f_{ij}\}$ be a set of generators for the ideal \mathfrak{a}_i . We can find a finitely generated \mathbb{F}_p -subalgebra $A \subset k$ such that $f_{ij} \in A[x_1, \dots, x_n]$ for all i and j . Let K be the fraction field of A and $S = K[x_1, \dots, x_n]$. Since R and S are free over R^{p^e} and S^{p^e} , respectively, Definition 2.6 and Proposition 2.1 imply that

$$\tau(\mathfrak{a}^c) = \tau((\mathfrak{a}_1 \cap S)^{c_1} \cdots (\mathfrak{a}_n \cap S)^{c_n})R,$$

where the test ideal on the right is computed in S . The result is then clear since S is essentially of finite type over \mathbb{F}_p . \square

Definition 3.17. The *constancy region* for a test ideal $\tau(\mathfrak{a}^c)$ is defined as the set of points $\mathbf{c}' \in \mathbb{R}_{\geq 0}^n$ such that $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^{\mathbf{c}'})$.

Lemma 3.18. $B^J(\mathfrak{a})$ consist of the points $\mathbf{c} \in \mathbb{R}_{\geq 0}^n$ such that $\tau(\mathfrak{a}^c) \not\subseteq J$.

Proof. Assume first that $\mathbf{c} = (\frac{\alpha_1}{p^e}, \dots, \frac{\alpha_n}{p^e})$ with $\alpha_i \in \mathbb{N}$. Choose a representation of \mathbf{c} with e large enough such that $\tau(\mathfrak{a}^c) = (\mathfrak{a}^\alpha)^{[1/p^e]}$. In this case we have

$$\mathbf{c} \in B^J(\mathfrak{a}) \iff \mathfrak{a}^\alpha \not\subseteq J^{[p^e]} \iff (\mathfrak{a}^\alpha)^{[1/p^e]} \not\subseteq J \iff \tau(\mathfrak{a}^c) \not\subseteq J.$$

For the general case, let $\mathbf{c} \in B^J(\mathfrak{a})$, this implies that $\mathbf{c} \in B^J(\mathfrak{a}, p^e)$ for some e . Therefore we can find $\mathbf{r} = (\frac{\alpha_1}{p^e}, \dots, \frac{\alpha_n}{p^e}) \in B^J(\mathfrak{a}, p^e) \subseteq B^J(\mathfrak{a})$, with $\alpha_i \in \mathbb{N}$, $\frac{\alpha_i}{p^e} \geq c_i$. By the first part this implies $\tau(\mathfrak{a}^{\mathbf{r}}) \not\subseteq J$, but as $\frac{\alpha_i}{p^e} \geq c_i$ for all i , we have that $\tau(\mathfrak{a}^{\mathbf{r}}) \subseteq \tau(\mathfrak{a}^c)$ hence $\tau(\mathfrak{a}^c) \not\subseteq J$.

For the reverse inclusion, let $\mathbf{c} \in \mathbb{R}_{\geq 0}^n$ be such that $\tau(\mathfrak{a}^c) \not\subseteq J$. By Proposition 3.10 there is a point $\mathbf{r} = (\frac{\alpha_1}{p^e}, \dots, \frac{\alpha_n}{p^e})$ with $\alpha_i \in \mathbb{N}$, $\frac{\alpha_i}{p^e} \geq c_i$ and $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^{\mathbf{r}})$, therefore $\tau(\mathfrak{a}^{\mathbf{r}}) \not\subseteq J$. We use the first part again and conclude $\mathbf{r} \in B^J(\mathfrak{a})$, but as $\frac{\alpha_i}{p^e} \geq c_i$ for all i , we deduce that $\mathbf{c} \in B^J(\mathfrak{a})$. \square

Theorem 3.19. If $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ are all contained in a maximal ideal \mathfrak{m} , then for each $\mathbf{c} \in \mathbb{R}_{\geq 0}^n$, there exist ideals I_1, \dots, I_d and J such that the constancy region for the test ideal $\tau(\mathfrak{a}^c)$ is given by $\bigcap_{i=1, \dots, d} B^{I_i}(\mathfrak{a}) \setminus B^J(\mathfrak{a})$.

Proof. We first show that this constancy region is bounded. As $\mathfrak{a}_i \subseteq \mathfrak{m}$ for all i we have that for any $\mathbf{c}' \in \mathbb{R}_{\geq 0}^n$ and e sufficiently large

$$\begin{aligned} \tau(\mathfrak{a}^{\mathbf{c}'}) &= (\mathfrak{a}_1^{[c'_1 p^e]} \cdots \mathfrak{a}_n^{[c'_n p^e]})^{[1/p^e]} \subseteq (\mathfrak{m}^{[c'_1 p^e] + \dots + [c'_n p^e]})^{[1/p^e]} \\ &\subseteq (\mathfrak{m}^{[c'_1 p^e + \dots + c'_n p^e] - n})^{[1/p^e]} \subseteq \mathfrak{m}^{[c'_1 + \dots + c'_n] - n + 1}. \end{aligned}$$

Since $\bigcap_s \mathfrak{m}^s = 0$, there is L such that $\tau(\mathfrak{a}^c) \not\subseteq \mathfrak{m}^L$, we deduce that for any \mathbf{c}' in the constancy region $\tau(\mathfrak{a}^{\mathbf{c}'}) = \tau(\mathfrak{a}^c) \not\subseteq \mathfrak{m}^L$, hence $c'_1 + \dots + c'_n \leq L$. This implies that the constancy region for $\tau(\mathfrak{a}^c)$ is bounded.

To deduce our description consider a sufficiently large hypercube $[\mathbf{0}, \mathbf{l}]$ containing the constancy region for $\tau(\mathfrak{a}^c)$. By Proposition 3.16, we know that the set $\mathcal{A} = \{\tau(\mathfrak{a}^c) \mid \mathbf{c} \in [\mathbf{0}, \mathbf{l}]\}$ is finite. Let I_1, \dots, I_d be the ideals in \mathcal{A} that are strictly contained in $\tau(\mathfrak{a}^c)$ and let $J = \tau(\mathfrak{a}^c)$. We claim that the constancy region for $\tau(\mathfrak{a}^c)$ is equal to $\bigcap_{i=1, \dots, d} B^{I_i}(\mathfrak{a}) \setminus B^J(\mathfrak{a})$. Lemma 3.18 implies that the set $\bigcap_{i=1, \dots, d} B^{I_i}(\mathfrak{a}) \setminus B^J(\mathfrak{a})$ is equal to the set of all \mathbf{r} such that $\tau(\mathfrak{a}^{\mathbf{r}}) \not\subseteq I_i$ for all i and $\tau(\mathfrak{a}^c) \subseteq \tau(\mathfrak{a}^{\mathbf{r}})$, or equivalently, $\tau(\mathfrak{a}^{\mathbf{r}}) = \tau(\mathfrak{a}^c)$ by our choice of I_i . \square

Remark 3.20. We can remove the condition that all ideals \mathfrak{a}_i are contained in a maximal ideal and still get a similar description. Explicitly, in each hypercube $[\mathbf{0}, \mathbf{l}]$ the constancy region is given by $\bigcap_{i=1, \dots, d} (B^{I_i}(\mathfrak{a}) \setminus B^J(\mathfrak{a})) \cap [\mathbf{0}, \mathbf{l}]$, for suitable I_1, \dots, I_d and J .

We now give a version of Skoda's theorem for mixed test ideals (see [BMS, Proposition 2.25] for the case of one ideal). This theorem allows us to describe the constancy regions in the first octant by describing only the constancy regions in a sufficiently large hypercube $[\mathbf{0}, \mathbf{l}] = [0, l_1] \times \dots \times [0, l_n]$.

Theorem 3.21. (*Skoda's Theorem*) *Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis for \mathbb{R}^n , and assume $1 \leq i \leq n$. If \mathfrak{a}_i is generated by m_i elements, then for every $\mathbf{s} = (s_1, \dots, s_n)$ with $s_i \geq m_i$, we have*

$$\tau(\mathfrak{a}^{\mathbf{s}}) = \mathfrak{a}^{\mathbf{e}_i} \tau(\mathfrak{a}^{\mathbf{s} - \mathbf{e}_i}).$$

Proof. We only need to prove $(\mathfrak{a}^{\lceil p^e \mathbf{s} \rceil})^{[1/p^e]} = \mathfrak{a}^{\mathbf{e}_i} (\mathfrak{a}^{\lceil p^e (\mathbf{s} - \mathbf{e}_i) \rceil})^{[1/p^e]}$ for e large enough.

Let $\mathbf{d} = (d_1, \dots, d_n)$ be a vector with integer coordinates and $d_i \geq p^e s_i$. We want to show that

$$(\mathfrak{a}^{\mathbf{d}})^{[1/p^e]} = \mathfrak{a}^{\mathbf{e}_i} (\mathfrak{a}^{\mathbf{d} - p^e \mathbf{e}_i})^{[1/p^e]},$$

from which the result follows.

Since $\mathfrak{a}^{\mathbf{d} - p^e \mathbf{e}_i} \cdot \mathfrak{a}_i^{[p^e]} \subseteq \mathfrak{a}^{\mathbf{d}} \subseteq ((\mathfrak{a}^{\mathbf{d}})^{[1/p^e]})^{[p^e]}$, we have

$$\mathfrak{a}^{\mathbf{d} - p^e \mathbf{e}_i} \subseteq (((\mathfrak{a}^{\mathbf{d}})^{[1/p^e]})^{[p^e]} : \mathfrak{a}_i^{[p^e]}) = ((\mathfrak{a}^{\mathbf{d}})^{[1/p^e]} : \mathfrak{a}_i)^{[p^e]},$$

where the equality is consequence of the flatness of Frobenius. Therefore

$$(\mathfrak{a}^{\mathbf{d} - p^e \mathbf{e}_i})^{[1/p^e]} \subseteq ((\mathfrak{a}^{\mathbf{d}})^{[1/p^e]} : \mathfrak{a}_i),$$

that is,

$$\mathfrak{a}^{\mathbf{e}_i} (\mathfrak{a}^{\mathbf{d} - p^e \mathbf{e}_i})^{[1/p^e]} \subseteq (\mathfrak{a}^{\mathbf{d}})^{[1/p^e]}.$$

For the reverse inclusion, note that since $d_i \geq m_i(p^e - 1) + 1$, in the product of d_i of the generators of \mathfrak{a}_i at least one should appear with multiplicity $\geq p^e$. Therefore $\mathfrak{a}^{\mathbf{d}} = \mathfrak{a}_i^{[p^e]} \cdot \mathfrak{a}^{\mathbf{d} - p^e \mathbf{e}_i}$, hence

$$\mathfrak{a}^{\mathbf{d}} \subseteq \mathfrak{a}_i^{[p^e]} \cdot \mathfrak{a}^{\mathbf{d} - p^e \mathbf{e}_i} \subseteq \mathfrak{a}_i^{[p^e]} \cdot ((\mathfrak{a}^{\mathbf{d} - p^e \mathbf{e}_i})^{[1/p^e]})^{[p^e]} = (\mathfrak{a}^{\mathbf{e}_i} \cdot (\mathfrak{a}^{\mathbf{d} - p^e \mathbf{e}_i})^{[1/p^e]})^{[p^e]},$$

which clearly implies $(\mathfrak{a}^{\mathbf{d}})^{[1/p^e]} \subseteq \mathfrak{a}^{\mathbf{e}_i} (\mathfrak{a}^{\mathbf{d} - p^e \mathbf{e}_i})^{[1/p^e]}$. \square

Proposition 3.22. *If c is an F -jumping number in the direction $\mathbf{r} = (r_1, \dots, r_n)$ then also cp is an F -jumping number in the direction \mathbf{r} .*

Proof. Note that $V_{\mathbf{r}}^I(\mathfrak{a}, p^{e+1}) = V_{\mathbf{r}}^{I^{[p]}}(\mathfrak{a}, p^e)$, hence $pC_{\mathbf{r}}^I(\mathfrak{a}) = C_{\mathbf{r}}^{I^{[p]}}(\mathfrak{a})$. \square

4. THE CONSTANCY REGIONS

In this section we prove our main result, Theorem 4.6 below. We begin by recalling our definition of p -fractals.

Let \mathcal{F} be the algebra of functions $\phi : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{Q}$. For each $q = p^e$ and every $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$ with $0 \leq b_i < q$ we define a family of operators $T_{q|\mathbf{b}} : \mathcal{F} \rightarrow \mathcal{F}$ by

$$T_{q|\mathbf{b}}\phi(t_1, \dots, t_n) = \phi((t_1 + b_1)/q, \dots, (t_n + b_n)/q).$$

Definition 4.1. Let $\phi : [0, l]^n \rightarrow \mathbb{Q}$ be a map and let denote also by ϕ its extension by zero to $\mathbb{R}_{\geq 0}^n$. We say that ϕ is a p -fractal if all the $T_{q|b}\phi$ span a finite dimensional \mathbb{Q} -subspace \bar{V} of \mathcal{F} . Furthermore, we say that an arbitrary $\phi \in \mathcal{F}$ is a p -fractal if its restriction to each hypercube $[0, l]$ is a p -fractal.

Remark 4.2. This definition is similar to the one in [MT, Definition 2.1]. The only difference is that in [MT, Definition 2.1] the domain of the functions is the hypercube $[0, 1] \times \dots \times [0, 1]$.

Recall that we are assuming R to be a regular, F -finite ring essentially of finite type over a field of characteristic $p > 0$, and $\mathfrak{a}_i \subseteq R$ are nonzero ideals.

Lemma 4.3. *Let $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}_{\geq 0}^n$, and $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be nonzero ideals of R then $\tau(\mathfrak{a}^{\mathbf{c}})^{[1/p^e]} = \tau(\mathfrak{a}^{\frac{1}{p^e}\mathbf{c}})$.*

Proof. Taking k large enough

$$\tau(\mathfrak{a}^{\mathbf{c}})^{[1/p^e]} = \left((\mathfrak{a}_1^{[c_1 p^k]} \dots \mathfrak{a}_n^{[c_n p^k]})^{[1/p^k]} \right)^{[1/p^e]}$$

and by [BMS, Lemma 2.4] the later contains

$$\begin{aligned} (\mathfrak{a}_1^{[c_1 p^k]} \dots \mathfrak{a}_n^{[c_n p^k]})^{[1/p^{k+e}]} &= (\mathfrak{a}_1^{\lceil \frac{c_1}{p^e} p^{k+e} \rceil} \dots \mathfrak{a}_n^{\lceil \frac{c_n}{p^e} p^{k+e} \rceil})^{[1/p^{k+e}]} \\ &= \tau(\mathfrak{a}^{\frac{1}{p^e}\mathbf{c}}). \end{aligned}$$

Therefore $\tau(\mathfrak{a}^{\mathbf{c}})^{[1/p^e]} \supseteq \tau(\mathfrak{a}^{\frac{1}{p^e}\mathbf{c}})$.

For the other inclusion note that

$$\begin{aligned} \tau(\mathfrak{a}^{\mathbf{c}}) &= (\mathfrak{a}_1^{[c_1 p^k]} \dots \mathfrak{a}_n^{[c_n p^k]})^{[1/p^k]} \\ &= (\mathfrak{a}_1^{\lceil \frac{c_1}{p^e} p^{k+e} \rceil} \dots \mathfrak{a}_n^{\lceil \frac{c_n}{p^e} p^{k+e} \rceil})^{[p^e/p^{k+e}]} \end{aligned}$$

that by [BMS, Lemma 2.4] is contained in

$$\left((\mathfrak{a}_1^{\lceil \frac{c_1}{p^e} p^{k+e} \rceil} \dots \mathfrak{a}_n^{\lceil \frac{c_n}{p^e} p^{k+e} \rceil})^{[1/p^{k+e}]} \right)^{[p^e]} = \tau(\mathfrak{a}^{\frac{1}{p^e}\mathbf{c}})^{[p^e]}$$

but this is equivalent to say

$$\tau(\mathfrak{a}^{\mathbf{c}})^{[1/p^e]} \subseteq \tau(\mathfrak{a}^{\frac{1}{p^e}\mathbf{c}}).$$

□

Lemma 4.4. *Let $\mathbf{l} = (l_1, \dots, l_n) \in \mathbb{Z}^n$ be such that l_i is the minimum number of generators of the ideal \mathfrak{a}_i . Let $\mathbf{b} \in \mathbb{Z}^n$ such that $l_i - 1 \leq b_i$. For all e , we have*

$$T_{p^e|\mathbf{b}}\chi_{\mathbf{a}}^I = T_{p^0|(\mathbf{l}-\mathbf{1})}\chi_{\mathbf{a}}^{(I^{[p^e]}:\mathfrak{a}^{\mathbf{b}-\mathbf{l}+1})},$$

where $\chi_{\mathbf{a}}^I$ denotes the characteristic function introduced in Definition 3.3.

Proof. We have that

$$T_{p^e|\mathbf{b}}\chi_{\mathbf{a}}^I(\mathbf{t}) = \chi_{\mathbf{a}}^I\left(\frac{1}{p^e}(\mathbf{t} + \mathbf{b})\right)$$

is equal to 1 if and only if, by Lemma 3.18, to

$$\tau(\mathfrak{a}^{\frac{1}{p^e}(\mathbf{t}+\mathbf{b})}) \not\subseteq I,$$

and by Lemma 4.3 this is

$$\tau(\mathfrak{a}^{\mathbf{t}+\mathbf{b}})^{[1/p^e]} \not\subseteq I,$$

but the later is equivalent to

$$\tau(\mathfrak{a}^{t+b}) \not\subseteq I^{[p^e]}.$$

As $b_i \geq l_i - 1$ by Skoda's Theorem the previous expresion becomes

$$\mathfrak{a}^{b-l+1} \cdot \tau(\mathfrak{a}^{t+l-1}) \not\subseteq I^{[p^e]}$$

Wich in turn is equivalent to

$$\tau(\mathfrak{a}^{t+l-1}) \not\subseteq (I^{[p^e]} : \mathfrak{a}^{b-l+1})$$

but this is the case if and only if

$$T_{p^0|(l-1)}\chi_{\mathfrak{a}}^{(I^{[p^e]}:\mathfrak{a}^{b-l+1})}(t) = \chi_{\mathfrak{a}}^{(I^{[p^e]}:\mathfrak{a}^{b-l+1})}(t+l-1)$$

is equal to 1

Note that a point of the form $\frac{1}{p^k}\mathbf{r} + (\mathbf{l} - \mathbf{1})$ with $\mathbf{r} \in \mathbb{Z}^n$ is in $B^{(I^{[p^e]}:\mathfrak{a}^{b-l+1})}(\mathfrak{a})$ if and only if $\mathfrak{a}^{\mathbf{r}+p^k(\mathbf{l}-\mathbf{1})} \not\subseteq (I^{[p^e]} : \mathfrak{a}^{b-l+1})^{[p^k]}$ if and only if $\mathfrak{a}^{\mathbf{r}} \cdot \mathfrak{a}^{p^k(\mathbf{l}-\mathbf{1})} \cdot (\mathfrak{a}^{b-l+1})^{[p^k]} \not\subseteq I^{[p^{e+k}]}$ this by Lemma 4.3 occurs if and only if $\mathfrak{a}^{\mathbf{r}+p^k\mathbf{b}} \not\subseteq I^{[p^{e+k}]}$, or equivalently $\frac{1}{p^{e+k}}\mathbf{r} + \frac{1}{p^e}\mathbf{b} \in B^I(\mathfrak{a})$. From this the result follows easily. \square

This lemma is especially useful when the ideals are principal, as we will see in the examples of Section 5.

Lemma 4.5. *In each hypercube $[\mathbf{0}, \mathbf{l}]$ there are only finitely many functions $\chi_{\mathfrak{a}}^I$. That is, the set $\{\chi_{\mathfrak{a}}^I|_{[\mathbf{0}, \mathbf{l}]}; I \subseteq R\}$ is finite.*

Proof. By Lemma 3.18, $B^I(\mathfrak{a})$ is the set of all points $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}_{\geq 0}^n$ such that $\tau(\mathfrak{a}^{\mathbf{c}}) \not\subseteq I$, hence $B^I(\mathfrak{a})$ is a union of constancy regions. By Lemma 3.16, we know that there are only finitely many constancy regions for bounded exponents, therefore there are only finitely many functions $\chi_{\mathfrak{a}}^I|_{[\mathbf{0}, \mathbf{l}]}$. \square

Theorem 4.6. There is a p -fractal function $\varphi : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{N}$ for which

$$\tau(\mathfrak{a}_1^{c_1} \dots \mathfrak{a}_n^{c_n}) = \tau(\mathfrak{a}_1^{d_1} \dots \mathfrak{a}_n^{d_n}) \iff \varphi(c_1, \dots, c_n) = \varphi(d_1, \dots, d_n),$$

and therefore the constancy regions are of the form $\varphi^{-1}(i)$ for some number i .

Proof. We first show that the functions $\chi_{\mathfrak{a}}^I$ are p -fractal. We want to prove that all the $T_{p^e|\mathbf{b}}\chi_{\mathfrak{a}}^I$ span a finite dimensional space. Lemma 4.4 states that all but finitely many of these functions have the form $T_{p^0|(\mathbf{l}-\mathbf{1})}\chi_{\mathfrak{a}}^J$ for different ideals J . Lemma 4.5 ensures that there are only finitely many of those in each hypercube $[\mathbf{0}, \mathbf{l}]$. From this it follows that $\chi_{\mathfrak{a}}^I$ is a p -fractal.

For $\mathbf{c} \in \mathbb{R}_{\geq 0}^n$, let $\eta_{\mathbf{c}}$ be the characteristic function associated to the constancy region $\tau(\mathfrak{a}^{\mathbf{c}})$. Remark 3.20 implies that that in each hypercube $[\mathbf{0}, \mathbf{l}]$, $\eta_{\mathbf{c}}|_{[\mathbf{0}, \mathbf{l}]} = (\chi_{\mathfrak{a}}^{I_1} \cdots \chi_{\mathfrak{a}}^{I_d} - \chi_{\mathfrak{a}}^J)|_{[\mathbf{0}, \mathbf{l}]}$ for some ideals I_1, \dots, I_d and J , therefore $\eta_{\mathbf{c}}$ is p -fractal.

Clearly there are countably many constancy regions, so we can numerate them. For every i , let $\mathbf{c}_i = (c_{i1}, \dots, c_{in})$ a point in the i -th constancy region, and we define

$$\varphi = \sum_{i \in \mathbb{N}} i \cdot \eta_{\mathbf{c}_i}.$$

This function satisfies the desired conditions. \square

Corollary 4.7. *Let $\eta_{\mathbf{c}}$ be the characteristic function associated to the constancy region of $\tau(\mathfrak{a}^{\mathbf{c}})$, then $\eta_{\mathbf{c}}$ is a p -fractal.*

5. AN EXAMPLE

In section 4 we showed that the characteristic functions of the constancy regions are p -fractal functions, Corollary 4.7. We use this fact and Proposition 2.1 to compute an explicit example. Throughout this section we use a subscript $*_p$ to denote that the number is written in base p . One of the main tools for computing examples is the following theorem:

Theorem 5.1. (*Lucas' Theorem [E]*) Fix non-negative integers $m \geq n \in \mathbb{N}$ and a prime number p . Write m and n in their base p expansions: $m = \sum_{j=0}^r m_j p^j$ and $n = \sum_{j=0}^r n_j p^j$. Then modulo p ,

$$\binom{m}{n} = \binom{m_0}{n_0} \cdot \binom{m_1}{n_1} \cdots \binom{m_r}{n_r},$$

where we interpret $\binom{a}{b}$ as zero if $a < b$. In particular, $\binom{m}{n}$ is non-zero mod p if and only if $m_j \geq n_j$ for all $j = 1, \dots, r$.

Remark 5.2. In particular if $m = p^k - 1$ all the coefficients in the expansion of $(x + y)^m$ are nonzero.

Example 5.3. (The Devil's Staircase) Let $R = \mathbb{F}_3[x, y]$, $f_1 = x + y$, and $f_2 = xy$. We want to describe the constancy regions for the test ideals $\tau(f^c)$.

We first show that there are five different test ideals in the region $[0, 1] \times [0, 1]$. More precisely, we show that

$$\tau(f^c) = \begin{cases} R \text{ or } (x, y), & \mathbf{c} \in [0, 1] \times [0, 1) \\ (x + y), & \mathbf{c} \in \{1\} \times [0, 1) \\ (xy), & \mathbf{c} \in [0, 1] \times \{1\} \\ (xy(x + y)), & \mathbf{c} = (1, 1). \end{cases}$$

We want to compute the test ideal at $(\frac{1}{3}, \frac{2}{3})$. By Proposition 3.12

$$\tau(f^{(0.13, 0.23)}) = \tau((f_1 \cdot f_2^2)^{\frac{1}{3}}).$$

By Proposition 2.3,

$$\tau((f_1 \cdot f_2^2)^{\frac{1}{3}}) = ((x + y)(xy)^2)^{[\frac{1}{3}]} = (x^3 y^2 + x^2 y^3)^{[\frac{1}{3}]}.$$

Finally, Proposition 2.1 gives

$$(x^3 y^2 + x^2 y^3)^{[\frac{1}{3}]} = (x, y),$$

and therefore

$$\tau(f_1^{c_1} \cdot f_2^{c_2}) \subseteq (x, y) \text{ if } c_1 \geq 1/3 \text{ and } c_2 \geq 2/3.$$

In particular, the test ideal associated to the points $(1 - \frac{1}{3^k}, 1 - \frac{1}{3^k})$ is contained in (x, y) . Now

$$\begin{aligned} \tau(f^{(1 - \frac{1}{3^k}, 1 - \frac{1}{3^k})}) &= ((x + y)^{3^k - 1} (xy)^{3^k - 1})^{[\frac{1}{3^k}]} \\ &= ((x^2 y + xy^2)^{3^k - 1})^{[\frac{1}{3^k}]} \end{aligned}$$

Since the terms $x^{2(3^k - 1)} y^{3^k - 1}$ and $x^{3^k - 1} y^{2(3^k - 1)}$ appear in the expansion of $(x^2 y + xy^2)^{3^k - 1}$ with nonzero coefficient, Remark 5.2. We conclude that $\tau(f^{(1 - \frac{1}{3^k}, 1 - \frac{1}{3^k})}) \supseteq (x, y)$. Therefore

$$\tau(f^{(1 - \frac{1}{3^k}, 1 - \frac{1}{3^k})}) = (x, y).$$

Thus there are only two test ideals in the region $[0, 1) \times [0, 1)$, these are R and (x, y) .

Clearly $\tau(f^{(1,0)}) = (x + y)$, and by Skoda's Theorem

$$\begin{aligned}\tau(f^{(1,1-\frac{1}{3^k})}) &= f_1 \cdot \tau(f^{(0,1-\frac{1}{3^k})}) \\ &= (x + y) \cdot ((xy)^{3^k-1})^{[\frac{1}{3^k}]} \\ &= (x + y),\end{aligned}$$

hence the only test ideal in the region $[0, 1) \times \{1\}$ is $(x + y)$.

In a similar way, $\tau(f^{(0,1)}) = (xy)$ and

$$\begin{aligned}\tau(f^{(1-\frac{1}{3^k},1)}) &= f_2 \cdot \tau(f^{(1-\frac{1}{3^k},0)}) \\ &= (xy) \cdot ((x + y)^{3^k-1})^{[\frac{1}{3^k}]} \\ &= (xy).\end{aligned}$$

Thus (xy) is the only test ideal that appears in the region $\{1\} \times [0, 1)$.

Lastly, note that the test ideal at $(1, 1)$ is

$$\tau(f^{(1,1)}) = ((x + y)xy).$$

We now show that $(\frac{1}{3}, \frac{2}{3})$ is a point in the boundary of $B^{(x,y)}(f)$ and then use the p -fractal structure to sketch the constancy regions.

For every k

$$\begin{aligned}\tau(f^{(\frac{1}{3}-\frac{1}{3^k}, \frac{2}{3}-\frac{1}{3^k})}) \\ = ((x + y)^{3^{k-1}-1}(xy)^{2 \cdot 3^{k-1}-1})^{[\frac{1}{3^k}]}.\end{aligned}$$

But in the expansion of $(x + y)^{3^{k-1}-1}$ every term appears with nonzero coefficient, Remark 5.2. In particular the term $(xy)^{\frac{3^{k-1}-1}{2}}(xy)^{2 \cdot 3^{k-1}-1}$ appears with non-zero coefficient when expanding the product $(x + y)^{3^{k-1}-1}(xy)^{2 \cdot 3^{k-1}-1}$. Since the degrees in x and y of this monomial are smaller than 3^k , by Proposition 2.1 we conclude that $\tau(f^{(\frac{1}{3}-\frac{1}{3^k}, \frac{2}{3}-\frac{1}{3^k})}) = R$. Thus

$$\chi_f^{(x,y)}(\frac{1}{3}, \frac{2}{3}) = 0$$

and

$$\chi_f^{(x,y)}([0, \frac{1}{3}) \times [0, \frac{2}{3})) = 1.$$

The later shows that the point $(\frac{1}{3}, \frac{2}{3})$ is in the boundary of constancy regions for R and (x, y) . We can use the p -fractal structure to find more points in this boundary. The idea is to break the region $[0, 1) \times [0, 1)$ into squares of length $1/3$ and find which of these must contain a boundary point. Then we apply the p -fractal structure to these squares to find the points.

For the points $(0, \frac{2}{3})$, $(\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, 1)$, and $(1, \frac{2}{3})$ we have:

$$\begin{aligned}\tau(f^{(0, \frac{2}{3})}) &= ((xy)^2)^{[\frac{1}{3}]} = R, \\ \tau(f^{(\frac{2}{3}, \frac{1}{3})}) &= ((x + y)^2 xy)^{[\frac{1}{3}]} = (x^3 y - x^2 y^2 + xy^3)^{[\frac{1}{3}]} = R\end{aligned}$$

and

$$\begin{aligned}\tau(f^{(\frac{1}{3}, 1)}) &= ((x + y)(xy)^3)^{[\frac{1}{3}]} = (xy) \subset (x, y), \\ \tau(f^{(1, \frac{2}{3})}) &= ((x + y)^3 x^2 y^2)^{[\frac{1}{3}]} = (x + y) \subset (x, y).\end{aligned}$$

Therefore there should be boundary points in the squares $[0, 1/3) \times [2/3, 1)$ and $[2/3, 1) \times [0, 1/3)$. Is easy to check that there are not boundary points in all the other

squares. From this and lemma 4.4 we know that $T_{3|(0,2)}\chi_f^{(x,y)} = T_{3|(2,1)}\chi_f^{(x,y)} = \chi_f^{(x,y)}$, since $\chi_f^{(x,y)}$ is the only characteristic function that is non constant in $[0, 1) \times [0, 1)$. Moreover,

$$\begin{aligned} \chi_f^{(x,y)}(0.01_3, 0.22_3) &= \chi_f^{(x,y)}(0_3 + 0.01_3, 0.2_3 + 0.02_3) \\ &= \chi_f^{(x,y)}\left(\frac{0_3 + 0.1_3}{3}, \frac{2_3 + 0.2_3}{3}\right) = T_{3|(0,2)}\chi_f^{(x,y)}(0.1_3, 0.2_3) \\ &= \chi_f^{(x,y)}(0.1_3, 0.2_3) = \chi_f^{(x,y)}\left(\frac{1}{3}, \frac{2}{3}\right) = 0 \end{aligned}$$

in a similar way

$$\chi_f^{(x,y)}(0.21_3, 0.12_3) = 0$$

and

$$\chi_f^{(x,y)}([0, 0.01_3) \times [0, 0.22_3)) = \chi_f^{(x,y)}([0, 0.21_3) \times [0, 0.12_3)) = 1.$$

This is the points $(0.01_3, 0.22_3)$ and $(0.21_3, 0.12_3)$ are also in the boundary. We can repeat the process by subdividing the squares $[0, 1/3) \times [2/3, 1)$ and $[2/3, 1) \times [0, 1/3)$ into smaller squares of length $1/9$ and obtain more points of the boundary. This process can be summarized as follows. Let A is the set of points obtained from $(0.1_3, 0.2_3)$ by successively applying the operations

$$(0.a_1 \dots a_n 1_3, 0.b_1 \dots b_n 2_3) \mapsto \begin{cases} (0.a_1 \dots a_n 01_3, 0.b_1 \dots b_n 22_3) \\ (0.a_1 \dots a_n 21_3, 0.b_1 \dots b_n 12_3) \end{cases}$$

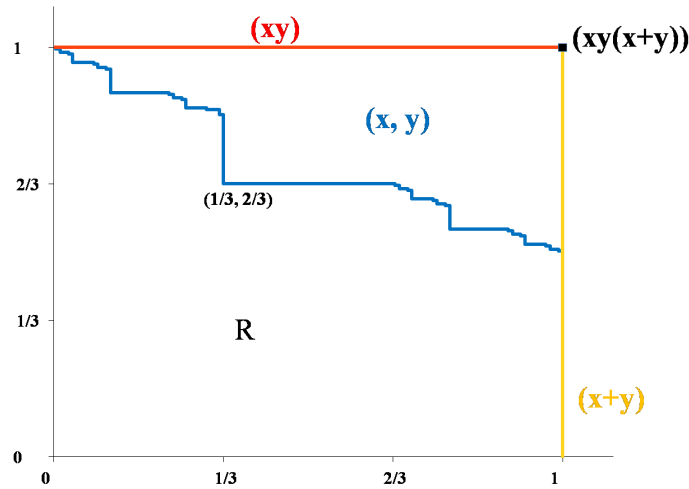
then

$$\chi_f^{(x,y)}(\mathbf{p}) = 0$$

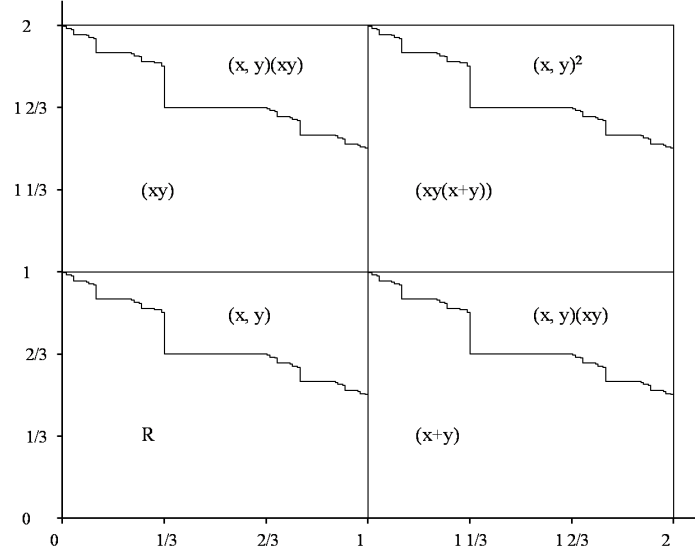
and

$$\chi_f^{(x,y)}([0, \mathbf{p})) = 1$$

for all $\mathbf{p} \in A$. This is, the points of A are points in the boundary. We can now sketch the regions of constancy in $[0, 1] \times [0, 1]$:



Using Skoda's theorem, we can describe the whole diagram of test ideals:



Remark 5.4. We choose the name Devil's Staircase for this example, because the resemblance to the Devil's Staircases or Cantor functions that appear in the basic courses of analysis.

Example 5.5. In a similar way, it can be shown that for any characteristic p the same polynomials give a staircase that has infinitely many steps. Indeed,

$$\tau(f^{(\frac{1}{p^k}, 1 - \frac{1}{p^k})}) = ((x+y)(xy)^{p^k-1})^{[\frac{1}{p^k}]} = (x, y)$$

but

$$\tau(f^{(\frac{2}{p^k}, 1 - \frac{2}{p^k})}) = ((x+y)^2(xy)^{p^k-2})^{[\frac{1}{p^k}]} = R$$

and so we have many different points in the line $x+2y=2$ with test ideal equal to (x, y) and infinitely many with test ideal equal to R . Therefore we can not expect that there are characteristics for which the region given by the test ideals will be the same as the one given by the multiplier ideals

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